WEAKLY STABLE BANACH SPACES

BY

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ABSTRACT

The class of stable Banach spaces, inspired by the stability theory in mathematical logic, was introduced by Krivine and Maurey and provided the proper context for the abstract formulation of Aldous' result of subspaces of L^1. In this paper we study the wider class of weakly stable Banach spaces, where the exchangeability of the iterated limits occurs only for sequences belonging to weakly compact subsets, introduced independently by Garling (in an earlier unpublished version of his expository paper on stable Banach spaces brought recently to our attention) and by the authors. Taking into account Rosenthal's application of the study of pointwise compact sets of Baire-1 functions (Rosenthal compact spaces) in the study of Banach spaces (for which l^1 does not embed isomorphically) and of the study of Rosenthal compact sets by Rosenthal and Bourgain–Fremlin–Talagrand, we prove the following analogue of the Krivine–Maurey theorem for weakly stable spaces: *If X is infinite dimensional and weakly stable then either l^p for some $p \geqq 1$ or c_0 embeds isomorphically in X* (§1). Garling (in the above reference) proved this result under the additional assumption that X^* is separable. We also construct an example of a Banach space X which is weakly stable, without an equivalent stable norm, and such that l^2 embeds isomorphically in every infinite dimensional subspace of X (§3).

§1

1.1. Definition. A separable Banach space X is called weakly stable if for every weakly compact subset K of X, every two sequences (x_n), (y_m) in K and every two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on the natural numbers the equality

$$\lim_{\substack{n\\ \mathcal{U}}} \lim_{\substack{m\\ \mathcal{V}}} \|x_n + y_m\| = \lim_{\substack{m\\ \mathcal{V}}} \lim_{\substack{n\\ \mathcal{U}}} \|x_n + y_m\|$$

holds.

This is a generalization of the notion of a stable Banach space, introduced by Krivine and Maurey in [10], in which the above equality holds for any two norm-bounded (and not necessarily weakly convergent) sequences (x_n), (y_m).

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Thus every stable space is weakly stable. It is also clear that every weakly stable, reflexive space is stable.

It is known that c_0 is not stable, and in fact that c_0 is not stable under any equivalent norm. It will be proved below (cf. Corollary 2.5) that c_0 and c are weakly stable. We note Garling has proved that c_0 is w-stable in an earlier version of [4], mentioned above.

On the other hand $c_0^{\omega^2}$ is not weakly stable (as remarked in 2.7 below).

The main result in this section is the following:

1.2. THEOREM. *If X is a weakly stable Banach space of infinite dimension, then for every $\varepsilon > 0$ X contains a subspace $(1+\varepsilon)$-isomorphic to l^p for some $1 \leqq p < \infty$ or to c_0.*

This theorem is a generalization of the celebrated theorem of Krivine–Maurey [10] for stable spaces, originally proved by Aldous [1] for $L^1[0,1]$ using probabilistic methods (random measures). The tools that we will use to prove our theorem are those introduced by Krivine and Maurey in [10] and the techniques of Rosenthal in [15]. We omit all details of proofs, where they are similar to those of the paper of Krivine–Maurey [10]; our proof of Theorem 1.2 is modeled after the proof given in Krivine [9].

We will need the following:

1.3. DEFINITIONS. Let X be a separable Banach space.

(a) A type on X is a function τ: $X \to \mathbf{R}^+$ for which there are a norm-bounded sequence (x_n) in X and an ultrafilter $\mathcal{U}$ on the set of natural numbers such that

$$\tau(x) = \lim_{\substack{n \\ \mathcal{U}}} \|x + x_n\| \qquad \text{for } x \in X.$$

(b) The set of all types on X is denoted by $\mathcal{T}(X)$; $\mathcal{T}(X)$ is a topological space, considered as a subspace of $(\mathbf{R}^+)^X$ with Cartesian (pointwise) topology.

(c) For $K \subset X$ we set

$$\mathcal{T}(X,K) = \Big\{ \tau \in \mathcal{T}(X)\text{: there are a sequence } x_n \in K \text{ and an ultrafilter } \mathcal{U} \text{ such that } \tau(x) = \lim_{\substack{n \\ \mathcal{U}}} \|x + x_n\| \text{ for } x \in X \Big\}.$$

(d) We set

$$\mathcal{T}_w(X) = \bigcup \{\mathcal{T}(X,K)\text{: } K \text{ weakly compact subset of } X\}.$$

An element of $\mathcal{T}_w(X)$ is called a weak type on X. Using the separability of X and

Eberlein's theorem it is easy, passing to subsequences if necessary, to dispense with the use of ultrafilters and to see that

$$\mathcal{T}_{\mathrm{w}}(X)=\left\{\tau\in\mathcal{T}(X)\text{: there are a sequence }(x_n)\subset X\text{ and }y\in X\text{ such that }\lim_n x_n=y\text{ weakly and }\tau(x)=\lim\|x_n+x\|\text{ for }x\in X\right\}.$$

We also set

$$\mathcal{T}_{\mathrm{wn}}(X)=\left\{\tau\in\mathcal{T}_{\mathrm{w}}(X)\text{: there are a sequence }(x_n)\subset X\text{ such that }\lim_n x_n=0\text{ weakly and }\tau(x)=\lim\|x_n+x\|\text{ for }x\in X\right\}.$$

An element of $\mathcal{T}_{\mathrm{wn}}(X)$ is called a weakly null type on X.

1.4. For $\tau\in\mathcal{T}(X)$ and $\lambda\in\mathbf{R}$, we recall the definition of $\lambda\tau\in\mathcal{T}(X)$, given by Krivine–Maurey [10]:

$$\lambda\tau=0\quad\text{if }\lambda=0,$$

$$(\lambda\tau)(x)=|\lambda|\,\tau\left(\frac{x}{\lambda}\right)\quad\text{if }\lambda\neq 0\text{ for }x\in X.$$

We note that if $\tau\in\mathcal{T}_{\mathrm{w}}(X)$, resp. $\mathcal{T}_{\mathrm{wn}}(X)$, then $\lambda\tau\in\mathcal{T}_{\mathrm{w}}(X)$, resp. $\mathcal{T}_{\mathrm{wn}}(X)$, for any $\lambda\in\mathbf{R}$.

1.5. Definitions. Let X be a weakly stable Banach space, and let $\sigma,\tau\in\mathcal{T}_{\mathrm{w}}(X)$.

We define $[\sigma,\tau]$ and $\sigma*\tau$. Since σ, $\tau\in\mathcal{T}_{\mathrm{w}}(X)$ there are weakly compact subsets K, L of X, sequences $(x_n)\subset K$, $(y_m)\subset L$, and ultrafilters $\mathcal{U}$, $\mathcal{V}$ on the set of natural numbers, such that

$$\sigma(x)=\lim_{\substack{n\\ \mathcal{U}}}\|x_n+x\|,\quad \tau(x)=\lim_{\substack{m\\ \mathcal{V}}}\|y_m+x\|\qquad\text{for }x\in X.$$

(a) We set

$$[\sigma,\tau]=\lim_{\substack{n\\ \mathcal{U}}}\lim_{\substack{m\\ \mathcal{V}}}\|x_n+y_m\|.$$

The fact that X is weakly stable implies that the operation

$$[\,.\,,\,.\,]\colon\mathcal{T}_{\mathrm{w}}(X)\times\mathcal{T}_{\mathrm{w}}(X)\to\mathbf{R}$$

is well-defined.

(b) For $y \in X$ we set

$$(\sigma * y)(x) = \sigma(y + x) \qquad \text{for } x \in X,$$

and generally we set

$$(\sigma * \tau) = \lim_{\substack{n \\ \mathcal{U}}} (\sigma * y_n),$$

where the limit is taken in the (pointwise) topology of $\mathcal{T}(X)$. The equality

$$(\sigma * \tau)(x) = \lim_{\substack{n \\ \mathcal{U}}} \lim_{\substack{m \\ \mathcal{V}}} \|x_n + y_m + x\| \qquad \text{for } x \in X$$

holds.

The fact that X is weakly stable implies that the operation

$$*: \mathcal{T}_w(X) \times \mathcal{T}_w(X) \rightarrow \mathcal{T}(X)$$

is well-defined.

1.6. Proposition. *Let X be a weakly stable Banach space.*

(a) *If $\sigma, \tau \in \mathcal{T}_w(X)$, then $\sigma * \tau \in \mathcal{T}_w(X)$.*

(b) *If $\sigma, \tau \in \mathcal{T}_{wn}(X)$, then $\sigma * \tau \in \mathcal{T}_{wn}(X)$.*

Proof. (a) Let (x_n), (y_n) be sequences in X, $x, y \in X$, such that weak-$\lim_n x_n = x$, weak-$\lim_n y_n = y$, $\sigma(z) = \lim_n \|z + x_n\|$, $\tau(x) = \lim_n \|z + y_n\|$ for $z \in X$. We choose a countable dense set $D = \{d_1, d_2, \ldots, d_k, \ldots\}$ in X. Inductively we construct a sequence $n_1 < n_2 < \cdots < n_l < \cdots$ of natural numbers such that

$$|(\sigma * \tau)(d_k) - \|x_{n_l} + y_{n_l} + d_k\|| \leqq 1/l \quad \text{for } k = 1, 2, \ldots, l \quad \text{and} \quad l = 1, 2, \ldots .$$

Then

$$(\sigma * \tau)(d_k) = \lim_l \|d_k + x_{n_l} + y_{n_l}\| \qquad \text{for } k = 1, 2, \ldots .$$

Since a type, in particular $\sigma * \tau$, is a uniformly continuous function on X, it follows that

$$(\sigma * \tau)(z) = \lim_l \|z + x_{n_l} + y_{n_l}\| \qquad \text{for } z \in X.$$

Of course weak-$\lim_l (x_{n_l} + y_{n_l}) = x + y$. Thus $\sigma * \tau \in \mathcal{T}_w(X)$.

(b) follows from the proof of (a).

1.7. Definition. A type $\sigma \in \mathcal{T}(X)$ is symmetric if $\sigma(x) = \sigma(-x)$ for $x \in X$. We denote by $\mathcal{T}_w^{\mathcal{S}}(X)$, $\mathcal{T}_{wn}^{\mathcal{S}}(X)$, the set of all weak and symmetric, weakly null and symmetric, respectively, types of X.

1.8. Definition. Let X be weakly stable, and $\sigma \in \mathcal{T}_w^{\mathcal{S}}(X)$. The spreading model of σ is a Banach space $Y \supset X$,

$$Y \text{ spanned by } X \cup \{\xi_k : k = 1, 2, \dots\},$$

and such that

$$\|x + \lambda_1\xi_1 + \cdots + \lambda_k\xi_k\| = (\lambda_1\sigma * \cdots * \lambda_k\sigma)(x) \qquad \text{for } k = 1, 2, \dots,$$
$$\lambda_1, \dots, \lambda_k \in \mathbf{R}, \qquad x \in X.$$

1.9. Lemma. *Let X be weakly stable and $\sigma \in \mathcal{T}_w^{\mathcal{S}}(X)$. Then the spreading model $Y = [X \cup \{\xi_k : k = 1, 2, \dots\}]$ of σ always exists, is unique up to isometry, and $\|x + \lambda_1\xi_1 + \cdots + \lambda_k\xi_k\|$ is invariant under permutations of the λ_i's and under change of λ_i to $\pm\lambda_i$. In particular, the sequence (ξ_k) is 1-unconditional.*

1.10. Definition. Let X be weakly stable and $\sigma \in \mathcal{T}_w^{\mathcal{S}}(X)$. Then σ is called an l^p-type, for some $p \geqq 1$, resp. c_0-type if $\alpha\sigma * \beta\sigma = (\alpha^p + \beta^p)^{1/p}\sigma$, resp. $\alpha\sigma * \beta\sigma = \max(\alpha, \beta)\sigma$, for $\alpha, \beta \geqq 0$.

1.11. Lemma. *Let X be weakly stable and $\sigma \in \mathcal{T}_w^{\mathcal{S}}(X)$, $\sigma \neq 0$. Then σ is an l^p-type for some $p \geqq 1$ or a c_0-type if and only if for all $\alpha \geqq 0$ there is $\beta \geqq 0$ such that $\sigma * \alpha\sigma = \beta\sigma$.*

(This is entirely analogous to Lemma III.1 in [10], making use of a result by Boehnenblust [2].)

1.12. Lemma. *Let X be weakly stable, $\sigma \in \mathcal{T}_w^{\mathcal{S}}(X)$, σ an l^p-type for some $p \geqq 1$, resp. a c_0-type, realized by the sequence (x_n) in X.*

Then there is a subsequence (y_n) of (x_n) equivalent to the usual basis of l^p, resp. of c_0. In fact,

$$\left(1 - \frac{1}{2^k}\right)\left(\sum_{n=k}^{\infty} |\lambda_n|^p\right)^{1/p} \leqq \left\|\sum_{n=k}^{\infty} \lambda_n y_n\right\| \leqq \left(1 + \frac{1}{2^k}\right)\left(\sum_{n=k}^{\infty} |\lambda_n|^p\right)^{1/p},$$

$$\textit{resp.} \quad \left(1 - \frac{1}{2^k}\right) \sup_{n \geqq k} |\lambda_n| \leqq \left\|\sum_{n=k}^{\infty} \lambda_n y_n\right\| \leqq \left(1 + \frac{1}{2^k}\right) \sup_{n \geqq k} |\lambda_n|,$$

for all sequences (λ_n) of real numbers eventually zero, and $k = 1, 2, \dots$.

(This is entirely analogous to Theorem III.1 in [10], and is based on a compactness argument using Ascoli's theorem.)

1.13. PROPOSITION. *If X is a separable Banach space and $l^1 \not\hookrightarrow X$, then $\mathcal{T}_{\mathrm{wn}}(X)$ is a closed subspace of $\mathcal{T}(X)$ (in its pointwise topology). In particular, $\mathcal{T}_{\mathrm{wn}}(X)$ and $\mathcal{T}^{\mathcal{S}}_{\mathrm{wn}}(X)$ are locally compact, σ-compact spaces.*

(This follows from Lemma 3.2 in Rosenthal [15].)

1.14. DEFINITION. Let X be weakly stable.

(a) A subset $\mathscr{C}$ of $\mathcal{T}^{\mathcal{S}}_{\mathrm{wn}}(X)$ is called a conic class if $\mathscr{C} \neq \emptyset$, $\mathscr{C} \neq \{0\}$, $\mathscr{C}$ is a closed subset of $\mathcal{T}^{\mathcal{S}}_{\mathrm{wn}}(X)$, $\lambda\sigma \in \mathscr{C}$ for $\sigma \in \mathscr{C}$ and $\lambda \in \mathbf{R}$ with $\lambda \geqq 0$, and $\sigma * \tau \in \mathscr{C}$ for $\sigma, \tau \in \mathscr{C}$.

(b) If $\mathscr{C}$ is a conic class in $\mathcal{T}^{\mathcal{S}}_{\mathrm{wn}}(X)$, $\sigma \in \mathscr{C}$, $\alpha, \beta > 0$ then σ is called $(\alpha, \beta, \mathscr{C})$-approximating type if for every $\varepsilon > 0$ and every neighborhood V of σ there is $\tau \in \mathscr{C} \cap V$ such that

$$|(\tau * \alpha\tau)(x) - (\beta\tau)(x)| \leqq \varepsilon \qquad \text{for } x \in X.$$

We set

$$\Gamma_{\alpha,\beta,\mathscr{C}} = \{\sigma \in \mathscr{C} : \sigma \text{ is } (\alpha, \beta, \mathscr{C})\text{-approximating type}\}.$$

1.15. COROLLARY. *Let X be weakly stable and $l^1 \not\hookrightarrow X$. Then every conic class contains a minimal conic class in $\mathcal{T}^{\mathcal{S}}_{\mathrm{wn}}(X)$.*

(Zorn's lemma argument, possible in presence of compactness.)

1.16. COROLLARY. *Let X be weakly stable $l^1 \not\hookrightarrow X$, let $\mathscr{C}$ be a conic class in $\mathcal{T}^{\mathcal{S}}_{\mathrm{wn}}(X)$ and $\alpha > 0$. Then there is $\beta > 0$ such that $\Gamma_{\alpha,\beta,\mathscr{C}} \neq \emptyset$, $\Gamma_{\alpha,\beta,\mathscr{C}} \neq \{0\}$.*

(The proof is analogous to that of Lemma IV. 4 in [10] and uses the symmetry of the sequence ξ_k (Lemma 1.12) in the spreading model of a weakly null, symmetric type and Proposition IV.1 in [10].)

1.17. LEMMA. *Let $(a_{i,j})_{i<j}$ be given real numbers for all $i < j$ and suppose that*

$$\lim_i \lim_j a_{ij} = a.$$

Then there is an increasing sequence $n(1) < n(2) < \cdots$ of positive integers so that

$$\lim_{\substack{i<j \\ i \to \infty}} a_{n(i),n(j)} = a.$$

(This is essentially a reformulation by Rosenthal [15] of Ramsey's partition principle in a form suitable for analysis.)

1.18. LEMMA. *Let X be a Banach space such that l^1 does not embed in X, and let $(k_{ij})_{i,j=1}^{\infty}$ be elements in X such that* $\lim_i \lim_j k_{ij} = 0$ *in the weak topology of X. Then there are $(i_n, j_n)_{n=1}^{\infty}$ in $i_n < j_n$ for all n and $i_n \to \infty$ as $n \to \infty$ such that* $\lim_n k_{i_n, j_n} = 0$ *in the weak topology of X.*

(This has been proved by Rosenthal [15], Lemma 3.3, using Ramsey's principle (Lemma 1.17 above) and the results for compact spaces of Baire-1 functions by Bourgain–Fremlin–Talagrand [3] and applications of such spaces on Banach spaces that do not contain isomorphically l^1 by Odell–Rosenthal [13].)

1.19. PROPOSITION. *Let X be weakly stable, such that $l^1 \not\hookrightarrow X$, and let $\sigma \in \mathcal{T}_w(X)$. Then the function*

$$\varphi_\sigma : \mathcal{T}_{wn}(X) \to \mathbf{R}$$

given by $\varphi_\sigma(\tau) = [\sigma, \tau]$ is continuous.

PROOF. Let (τ_n) be a sequence in $\mathcal{T}_{wn}(X)$, $\tau \in \mathcal{T}_{wn}(X)$, and $\tau_n \to \tau$ pointwise, and we wish to prove that $[\sigma, \tau_n] \to [\sigma, \tau]$. It is enough to prove that for every subsequence (τ_{n_k}) of (τ_n) there is a further subsequence $(\tau_{n_{k_l}})$ such that $[\sigma, \tau_{n_{k_l}}] \to [\sigma, \tau]$.

Without loss of generality assume that (τ_{n_k}) is the original sequence (τ_n). Since $\tau_n \in \mathcal{T}_{wn}(X)$, there is a sequence $(y_j^n)_{j=1}^{\infty}$ in X such that $\lim_j y_j^n = 0$ weakly in X and $\tau_n(x) = \lim_j \| y_j^n + x \|$ for $x \in X$ for all $n = 1, 2, \ldots$. We then have that

$$\lim_j \sigma(y_j^n) = [\sigma, \tau_n] \qquad \text{for } n = 1, 2, \ldots .$$

We may assume that

$$(1) \qquad |[\sigma, \tau_n] - \sigma(y_j^n) \leqq 1/n \qquad \text{for } j = 1, 2, \ldots, \quad n = 1, 2, \ldots .$$

We also have

$$\tau(x) = \lim_n \tau_n(x) = \lim_n \lim_j \| y_j^n + x \| \qquad \text{for } x \in X.$$

From Lemma 1.17 (Ramsey's principle) and the separability of X there is an increasing sequence $m(1) < m(2) < \cdots$ such that

$$\tau(x) = \lim_{\substack{i<j \\ i \to \infty}} \| x + y_{m(j)}^{m(i)} \| \qquad \text{for } x \in X.$$

It is clear that $\lim_i \lim_j y_{m(j)}^{m(i)} = 0$ weakly in X. It now follows from Rosenthal's Lemma 1.18 that there is a sequence $(i_l, j_l)_{l=1}^{\infty}$ with $i_l < j_l$ for $l = 1, 2, \ldots$ and $i_l \to \infty$ as $l \to \infty$ such that $\lim_l y_{m(j_l)}^{m(i_l)} = 0$ weakly in X. It is then clear that we have

$$\tau(x) = \lim_l \|x + y_{m(j_l)}^{m(i_l)}\| \qquad \text{for } x \in X,$$

hence

$$[\sigma, \tau] = \lim_l \sigma(y_{m(j_l)}^{m(i_l)}).$$

From (1) we also have

$$|[\sigma, \tau_{m(i_l)}] - \sigma(y_{m(j_l)}^{m(i_l)})| \leqq \frac{1}{m(i_l)},$$

hence finally $\lim_l [\sigma, \tau_{(m(i_l))}] = [\sigma, \tau]$, as required. The proof of the proposition is complete.

1.20. Remark. For $x \in X$ we set

$$\mathcal{T}_{wn}(x) \left\{ \sigma \in \mathcal{T}(X)\text{: there is a sequence } (x_n) \text{ in } X \text{ such that } \text{weak-}\lim_n x_n = x \text{ and } \sigma(z) = \lim_n \|z + x_n\| \text{ for } z \in X \right\}.$$

The following statement is proved exactly as Proposition 1.19:
Let X be weakly stable, $l^1 \not\hookrightarrow X$, $\sigma \in \mathcal{T}_w(X)$, and $x \in X$. Then the function

$$\varphi_\sigma : \mathcal{T}_{wx}(X) \to \mathbf{R}$$

given by $\varphi_\sigma(\tau) = [\sigma, \tau]$ is continuous.

1.21. Proposition. *Let X be weakly stable, and $l^1 \not\hookrightarrow X$. Then the convolution function*

$$*: \mathcal{T}_{wn}(X) \times \mathcal{T}_{wn}(X) \to \mathcal{T}_{wn}(X)$$

is separately continuous.

(Immediate from Remark 1.20.)

We have two consequences of this proposition.

1.22. Corollary. *Let X be weakly stable, $l^1 \not\hookrightarrow X$, and let $\mathcal{C}$ be a conic class in $\mathcal{T}_{wn}^{\mathcal{S}}(X)$ such that there are $\alpha, \beta > 0$ and $\sigma \in \mathcal{C}$, $\sigma \neq 0$, σ $(\alpha, \beta, \mathcal{C})$-approximating type. Then $\Gamma_{\alpha,\beta,\mathcal{C}}$ is a conic class.*

(It follows from Proposition 1.21; cf. Lemma IV.5 in [10].)

1.23. COROLLARY. *Let X be weakly stable, $l^1 \not\hookrightarrow X$, and let $\mathscr{C}$ be a conic class in $\mathscr{T}^{\mathscr{S}}_{\mathrm{wn}}(X)$. Then there exists a dense set D of $\mathscr{C}$ such that every $(\sigma, \tau) \in D \times \mathscr{C}$ is a point of continuity of the convolution*

$$*: \mathscr{C} \times \mathscr{C} \to \mathscr{C}.$$

(This is an immediate consequence of Proposition 1.21 and Namioka's theorem [11] on separately continuous real-valued functions: recall that $\mathscr{C}$ is a locally compact and σ-compact space.)

1.24. COROLLARY. *Let X be weakly stable, $l^1 \not\hookrightarrow X$, $\mathscr{C}$ a minimal conic class of $\mathscr{T}^{\mathscr{S}}_{\mathrm{wn}}(X)$ and $\alpha > 0$. Then there is $\beta > 0$ such that $\mathscr{C} = \Gamma_{\alpha,\beta,\mathscr{C}}$.*

(Immediate from Corollaries 1.16 and 1.22.)

1.25. COROLLARY. *Let X be weakly stable, $l^1 \not\hookrightarrow X$, and $\mathscr{C}$ a conic class in $\mathscr{T}^{\mathscr{S}}_{\mathrm{wn}}(X)$. Then $\mathscr{C}$ contains an l^p-type for some $p > 1$ or a c_0-type.*

(It follows from Corollaries 1.15, 1.24, 1.23 and Lemma 1.11.)

PROOF OF THEOREM 1.2. Immediate from Corollary 1.25 and Lemma 1.12, and James' result [8], stating that if $l^1 \hookrightarrow X$ and $\varepsilon > 0$ then there is a $(1+\varepsilon)$-embedding of l^1 into X.

1.26. REMARK. Recall that a Banach space X has Schur's property if every weakly convergent sequence in X is norm convergent. It follows immediately from the definitions that

every Banach space that has Schur's property is weakly stable.

It follows from Theorem 1.2 that if X has Schur's property then $l^1 \hookrightarrow X$. (This is a result of Rosenthal in [14].)

§2. Krivine and Maurey have proved in [10] (Theorem II.1) that the l_p-sum $(1 \leqq p < +\infty)$ of stable Banach spaces is stable. In the same way, we can prove that the l_p-sum of weakly stable Banach spaces is weakly stable.

In the present section we prove that the c_0-sum of weakly stable Banach spaces need not be weakly stable. We state a general theorem of interchange of double limits (whose proof is similar to Theorem 5 in [12], and is therefore omitted), and then we establish a condition for the preservation of the property of weakly stable Banach space in c_0-sums.

2.1. PROPOSITION. *Let X be a weakly stable Banach space, and we set $Y = (\bigoplus \Sigma_{i \in \mathbf{N}} X_i)_0$, with $X_i = X$ for $i \in \mathbf{N}$. If Y is weakly stable then every non-zero weakly null symmetric type of X is a c_0-type.*

PROOF. Let $\tau \in \mathcal{T}^{\mathcal{S}}_{\mathrm{wn}}(X) \backslash \{0\}$, and let $(x_n)_{n \in \mathbf{N}}$ be a sequence in X such that w-$\lim_n x_n = 0$ and $\tau(x) = \lim_n \|x_n + x\|$. We assume, without loss of generality, that $\tau(0) = 1$.

Let $x \in X$. We will prove that $(\alpha\tau * \beta\tau)(x) = (\max\{\alpha, \beta\} \cdot \tau)(x)$ for $\alpha, \beta \geqq 0$. We first assume that $\alpha \neq 0$, say $\alpha > \beta \geqq 0$. We define two sequences $(y_l)_{l \in \mathbf{N}}$, $(z_k)_{k \in \mathbf{N}}$ as follows:

$$y_l(i) = 0 \quad \text{for } i \neq l, \qquad \text{and}$$

$$y_l(l) = \alpha x_l + x; \qquad \text{and}$$

$$z_k(i) = \beta x_k \quad \text{for } i \leqq k,$$

$$= 0 \qquad \text{for } i > k.$$

It is clear that w-$\lim_l y_l =$ w-$\lim_k z_k = 0$. We also have

$$\lim_l \lim_k \|y_l + z_k\| = \max\{\beta, (\alpha\tau * \beta\tau)(x)\}, \qquad \text{and}$$

$$\lim_k \lim_l \|y_l + z_k\| = \max\{\beta, (\alpha\tau)(x)\}.$$

Since Y is weakly stable, we have

$$\max\{\beta, (\alpha\tau * \beta\tau)(x)\} = \max\{\beta, (\alpha\tau)(x)\}. \tag{1}$$

Furthermore,

$$(\alpha\tau)(x) > \beta. \tag{2}$$

In fact, if $(\alpha\tau)(x) \leqq \beta$, then $(\alpha\tau)(-x) \leqq \beta$, since τ is symmetric, hence $\lim_n \|\alpha x_n + x\| \leqq \beta$, $\lim_n \|\alpha x_n - x\| \leqq \beta$; thus $\alpha \leqq \beta$, a contradiction.

From (1) and (2) we have that

$$(\alpha\tau * \beta\tau)(x) = (\alpha\tau)(x) = (\max(\alpha, \beta) \cdot \tau)(x).$$

Finally, if $\alpha = \beta$, then we set $\beta_n = \alpha - 1/n$, $n \in \mathbf{N}$, and we have

$$(\alpha\tau * \beta_n\tau)(x) = (\alpha\tau)(x) \qquad \text{for } n \in \mathbf{N},$$

hence

$$(\alpha\tau * \alpha\tau)(x) = (\alpha\tau)(x).$$

The proof of the proposition is complete.

2.2. COROLLARY. *The space* $Y = (\bigoplus \Sigma_{i \in \mathbf{N}} X_i)_0$, *with* $X_i = l^p$ *for* $i \in \mathbf{N}$, *is not weakly stable, if* $p > 1$.

2.3. THEOREM. *Let* $(X_i)_{i \in I}$ *be a family of weakly stable Banach spaces, set* $X = (\bigoplus \Sigma_{i \in I} X_i)_0$, $Z = (\bigoplus \Sigma_{i \in I} X_i)_\infty$, *let* $u \in Z \setminus X$, *set* Y *to be the linear span in* Z *of* $X \cup \{u\}$, *let* K *be a weakly compact subset of* Y, $y_n, z_n \in X$, $c_n, d_n \in \mathbf{R}$ *with* $c_n u + y_n$, $d_n u + z_n \in K$ *for* $n = 1, 2, \ldots$, $\mathcal{U}$ *and* $\mathcal{V}$ *ultrafilters on the set of natural numbers, set* $s_n = (\| y_n(i) \|)_{i \in I}$, $t_n = (\| z_n(i) \|)_{i \in I}$ *and assume that*

$$\text{weak-}\lim_{\substack{n \\ \mathcal{U}}} s_n,\ \text{weak-}\lim_{\substack{n \\ \mathcal{V}}} t_n \in c_0(I).$$

Then

$$\lim_{\substack{n \\ \mathcal{U}}} \lim_{\substack{m \\ \mathcal{V}}} \| c_n u + y_m + d_m u + z_m \|_\infty = \lim_{\substack{m \\ \mathcal{V}}} \lim_{\substack{n \\ \mathcal{U}}} \| c_n u + y_n + d_m u + z_m \|_\infty.$$

Theorem 2.3 is a generalization of Theorem 5 in [12], which forms the essential part of the proof of non-existence of separable stable spaces containing isomorphic copies of all separable stable Banach spaces. (This is a result proved independently, in a different manner, by Guerre [5] as well.)

As special cases we get immediately.

2.4. COROLLARY. *Let* $(X_i)_{i \in I}$, X, Y *and* Z *be as in Theorem* 2.1. *If* X_i *has Schur's property for* $i \in I$, *then* Y *is weakly stable.*

2.5. COROLLARY. c_0, c *are weakly stable Banach spaces.*

REMARK. The following remark is contained in an earlier version of [4]: For every $\varepsilon > 0$ there is a norm $|||\cdot|||$ on c_0, $(1+\varepsilon)$-equivalent with the usual norm $\|\cdot\|_0$ on c_0, so that $(c_0, |||\cdot|||)$ is not weakly stable. In fact, for $x = (x_n) \in c_0$, set

$$|||x||| = \|x\|_0 + \frac{\varepsilon}{2} \max\{|x_{2k}| + |x_{2l-1}| : k < l\}.$$

If we denote by (e_n) the usual basis of c_0, then

$$\lim_k \lim_l \| e_{2k} + e_{2l+1} \| = 1 + \varepsilon,$$

$$\lim_l \lim_k \| e_{2k} + e_{2l+1} \| = 1 + \varepsilon/2.$$

2.6. PROPOSITION. *Let* X *be a separable Banach space, such that* $c_0 \not\hookrightarrow X$, *and* *set* $Y = (\bigoplus \Sigma_{i \in \mathbf{N}} X_i)_0$, *with* $X_i = X$ *for* $i \in \mathbf{N}$. *The following two statements are equivalent*:

(i) Y *is weakly stable*;

(ii) X *has Schur's property.*

PROOF. (i) $\Rightarrow$ (ii). If X does not have Schur's property, then there is a non-zero, weakly null symmetric type on X. By Proposition 2.1, this type is a c_0-type. Hence, by Lemma 1.2, c_0 embeds isomorphically in X.

(ii) $\Rightarrow$ (i). By Corollary 2.3, Y is weakly stable.

2.7. EXAMPLE. We will show that the space $c_0^{\omega^2}$ of all real valued continuous functions on the ordinal space ω^2 vanishing at ∞ is not weakly stable. Since

$$c_0^{\omega^2} = \left(\bigoplus \sum_{n=1}^{\infty} X_n \right)_0,$$

where X_n is isometrically identified with the space of all continuous functions on $\{\alpha; (n-1)\omega < \alpha \leqq n\omega\}$, in turn isometric to the weakly stable space c for $n = 1, 2, \ldots$, it follows that Theorem 2.1 cannot be improved significantly.

We define sequences $(x_l)_{l=1}^{\infty}$, $(y_k)_{k=1}^{\infty}$ in $c_0^{\omega^2}$ as follows:

$$\begin{aligned} (x_l(n))(\alpha) &= 0 \quad \text{if } n \leqq 1 \quad \text{and} \quad \alpha \neq (n-1)\omega + l, \\ &= 1 \quad \text{if } n \leqq l \quad \text{and} \quad \alpha = (n-1)\omega + l, \\ &= 0 \quad \text{if } n > l; \end{aligned}$$

$$\begin{aligned} y_k(n)(\alpha) &= 0 \quad \text{if } n \leqq k \quad \text{and} \quad (n-1)\omega < \alpha < (n-1)\omega + k, \\ &= 1 \quad \text{if } n \leqq k \quad \text{and} \quad (n-1)\omega + k \leqq \alpha \leqq n\omega, \\ &= 0 \quad \text{if } n > k. \end{aligned}$$

Then $\lim_l x_l(n)\ (\alpha) = 0 = \lim_k y_k(n)(\alpha)$ for every $(n-1)\omega < \alpha \leqq n\omega$, $n = 1, 2, \ldots$, hence $x_l \to 0$, $y_k \to 0$ weakly in $c_0^{\omega^2}$.

We note that

$$\begin{aligned} \| x_l + y_k \| &= 1 \quad \text{if } l < k, \\ &= 2 \quad \text{if } l > k. \end{aligned}$$

Hence, $\lim_l \lim_k \| x_l + y_k \| = 1 \neq 2 = \lim_k \lim_l \| x_l + y_k \|$.

§3

AN EXAMPLE. In this section we will prove the existence of an infinite dimensional Banach space X which (a) is weakly stable, (b) does not admit an equivalent stable norm (hence it is not reflexive), and (c) for every $\varepsilon > 0$ l^2 embeds $(1+\varepsilon)$-isomorphically in every infinite dimensional subspace of X.

The space X has a norm that is a variant of the norm of James' space, given in [7]. The existence of a space X with these properties shows that the concept of weakly stable Banach spaces is a proper generalization of the concept of stable spaces, even in the absence of any embedding of c_0.

3.1. *Preliminaries on Complete Minimal Systems.* Given a sequence (x_n) in a normed space X, we denote by $[(x_n)]$ its closed linear span.

A sequence (x_n) in a Banach space X is a complete minimal system if

$$X = [(x_n)], \quad \text{and}$$

$$x_n \notin [(x_m)_{m=1, m\neq n}^{\infty}] \quad \text{for } n = 1, 2, \ldots .$$

The Hahn–Banach theorem implies that a sequence (x_n) is a complete minimal system for the space X if and only if $[(x_n)] = X$, and there is $(x_n^*) \subset X^*$ such that $x_m^*(x_n) = \delta_{n,m}$ for $n, m = 1, 2, \ldots$. A sequence $((x_n, x_n^*))$ as above is said to be a biorthogonal system. For such a system $((x_n, x_n^*))$ and for $m \in \mathbf{N}$ we define $P_m : X \to X$ by $P_m(x) = \sum_{i=1}^m x_i^*(x) x_i$.

It is clear that P_m defines a bounded projection on the space $[(x_i)_{i=1}^m]$ and that $\operatorname{Ker} P_m = [(x_n)_{n=m+1}^{\infty}] = X_m$ for $m = 1, 2, \ldots$.

Hence, an element $x \in X$ belongs to X_m if and only if $x_i^*(x) = 0$ for all $i \leqq m$.

Using the above remarks it is easy to prove the following

3.2. Proposition. *Let (x_n) be a complete minimal system for a Banach space X. Then the following hold:*

(a) *If (z_n) is a sequence in X equivalent to the usual basis of l^p, $1 \leqq p < \infty$, or of c_0, then there exists a sequence (w_n) of blocks of (x_n) which is also equivalent to the basis of l^p, $1 \leqq p < \infty$, or of c_0, respectively.*

(b) *If (z_n) converges weakly to 0, then there exists a subsequence $(z_{n_k})_{k=1}^{\infty}$ of (z_n) and a sequence of blocks $(y_{n_k})_{k=1}^{\infty}$ of (x_n) such that*

$$\lim_k \| y_{n_k} - z_{n_k} \| = 0.$$

(The simple proof of Proposition 3.2 is omitted.)

3.3. *Definition of the Norm of the Space X.* We denote by Y the linear space of all real sequences $(x(0) = 0, x(1), \ldots, x(n), \ldots)$ that are eventually zero.

For $x \in Y$ we define

$$\|x\| = \sup_{p_1 < p_2 < \cdots < p_k} \left(\sum_{i=1}^{k-1} \frac{(x(p_{i+1}) - x(p_i))^2}{p_{i+1} - p_i} \right)^{1/2}.$$

It is clear that $\|\cdot\|$ defines a norm on Y and we denote by X the completion of the normed space $(Y, \|\cdot\|)$.

We denote by e_n, $n = 1, 2, \ldots$ the vectors of Y which vanish everywhere except for the nth term, where $e_n(n) = 1$. As we will show, the sequence (e_n) is a complete minimal system for X, which however fails to be a basis.

Let us note that for every $n = 1, 2, \ldots$ the vector $z_n = \Sigma_{i=1}^n e_i$ is easily seen to be of norm equal to $\sqrt{2}$, hence the sequence (z_n) is norm-bounded.

3.4. Proposition. *For $n = 1, 2, \ldots$, we define the linear functional e_n^*: $Y \to \mathbf{R}$ with the rule $e_n^*(x) = x(n)$. Then e_n^* is bounded and in fact $\|e_n^*\| = n^{1/2}$.*

Proof. We notice first that if $x \in Y$ and $\|x\| \leqq 1$, then $|x(n)| \leqq n^{1/2}$. To see this we consider the partition $0 = p_1 < p_2 = n$; it follows that

$$\left(\frac{x(n)^2}{n} \right)^{1/2} \leqq \|x\| \leqq 1,$$

hence

$$|x(n)| \leqq n^{1/2}.$$

Thus the functional e_n^* is bounded and $\|e_n^*\| \leqq n^{1/2}$.

The converse inequality follows immediately from the next lemma (which incidentally is not needed for the proof of the claimed properties for X).

3.5. Lemma. *For all $\varepsilon > 0$ and $n \in \mathbf{N}$ there exists $x_{n,\varepsilon} \in Y$ such that*

$$\|x_{n,\varepsilon}\| \leqq 1/n^{1/2} + \varepsilon \quad \textit{and} \quad x_{n,\varepsilon}(n) = 1.$$

Proof. We choose $k \in \mathbf{N}$ so that

$$\left(\frac{1}{n} + \frac{1}{k} \right)^{1/2} \leqq \frac{1}{n^{1/2}} + \varepsilon.$$

Next we note that there is $m \in \mathbf{N}$ such that for $l_1 \leqq n$ and $l_2 \leqq k$, the following inequality holds:

$$(*) \qquad \frac{\left(\frac{l_1}{n} - \frac{l_2}{k} \right)^2}{m^{1/2}} < \frac{(n - l_1)^2}{n^2 l_1^{1/2}} + \frac{(k - l_2)^2}{k^2 l_2^{1/2}}.$$

We define now the following element $x_{n,\varepsilon}$ of Y:

$$\begin{aligned} x_{n,\varepsilon}(l) &= \frac{l}{n} \quad \text{for } l = 0,1,2,\ldots,n, \\ &= 1 \quad \text{for } l = n+1, \\ &= \frac{k-l+(n+1)}{k} \quad \text{for } l = n+2,\ldots,n+k, \\ &= 0 \quad \text{for } l > n+k. \end{aligned}$$

CLAIM. $\|x_{n,\varepsilon}\| \leqq (1/n + 1/k)^{1/2}$.

Indeed, choose $p_1 < p_2 < \cdots < p_s$ a finite sequence, so that

$$\|x_{n,\varepsilon}\| = \left(\sum_{i=1}^{s-1} \frac{(x_{n,\varepsilon}(p_{i+1}) - x_{n,\varepsilon}(p_i))^2}{p_{i+1} - p_i} \right)^{1/2}.$$

We note that there is no $i < s$ so that $p_i \leqq n$ and $p_{i+1} > n+1$. Next let us denote by i_0, j_0 the greatest index so that $p_{i_0} \leqq n$ and the smallest index with $n+1 \leqq p_{j_0}$ respectively. Then since the partition $p_1 < p_2 < \cdots < p_s$ realizes the norm of $x_{n,\varepsilon}$, we must have that $p_{i_0} = n$ and $p_{j_0} = n+1$.

Hence, the sequence $p_1 < p_2 < \cdots < p_s$ has the form

$$p_1 < p_2 < \cdots < p_{i_0} = n < p_{i_0+1} < \cdots < p_{j_0} = n+m < \cdots < p_s,$$

and

$$\|x_{n,\varepsilon}\| = \left(\sum_{i=1}^{i_0-1} \frac{(p_{i+1} - p_i)}{n^2} + \sum_{j=n+m}^{s-1} \frac{(p_{j+1} - p_j)}{k^2} \right)^{1/2}.$$

Finally we get that

$$\|x_{n,\varepsilon}\| \leqq \left(\frac{1}{n} + \frac{1}{k} \right)^{1/2},$$

proving the claim.

The proofs of Lemma 3.5 and of Proposition 3.4 are now complete.

3.6. PROPOSITION. *The space X is not reflexive.*

PROOF. Consider the sequence (z_n), where $z_n = \sum_{i=1}^{n} e_i$. As we have noticed in 3.3 this sequence is norm bounded. We claim that no subsequence of (z_n) is weakly convergent. Assume on the contrary that there exist $x \in X$ and a subsequence (z_{n_k}) such that

$$\text{w-}\lim_k z_{n_k} = x.$$

We remark that for all $n = 1, 2, \ldots$ $e_n^*(x) = 1$ and since $\| e_1^* \| = 1$, we get $\| x \| \geqq 1$. Choose $x_1 \in X$ such that $\| x - x_1 \| < \frac{1}{4}$ and $x_1 = \sum_{i=1}^m \alpha_i e_i$ with $\alpha_i \neq 0$ and $\alpha_i \neq \alpha_j$ for every $i \neq j$, $1 \leqq i, j \leqq m$. Since $1 \leqq \| x \| \leqq \sqrt{2}$ we have $\frac{3}{4} \leqq \| x_1 \| \leqq \sqrt{2} + \frac{1}{4}$. Also, for all $1 \leqq i \leqq m$,

$$|\alpha_i| \leqq \sqrt{i}(\sqrt{2} + \tfrac{1}{4}) \leqq \sqrt{m}(\sqrt{2} + \tfrac{1}{4}).$$

Let $p_1 < p_2 < \cdots < p_k \leqq m + 1$ be the partition that realizes the norm of the element x_1, i.e.

$$\| x_1 \| = \left(\sum_{i=1}^{k-1} \frac{(\alpha_{p_{i+1}} - \alpha_{p_i})^2}{p_{i+1} - p_i} \right)^{1/2}.$$

We choose $0 < \varepsilon < \min(\frac{1}{4}, \frac{1}{2} | \alpha_{p_{i+1}} - \alpha_{p_i} | : 1 \leqq i \leqq k - 1)$ so that

$$(\| x_1 \|^2 - 16 \sqrt{m} \varepsilon (m+1))^{1/2} > \| x_1 \| - \tfrac{1}{4},$$

and we choose $x_2 \in X$ such that $\| x - x_2 \| < \varepsilon / \sqrt{m+1}$ and $x_2 = \sum_{i=1}^{m_1} \beta_i e_i$. Then

$$\| x_1 - x_2 \| \leqq \| x - x_1 \| + \| x - x_2 \| < \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

We remark that

$$| e_{p_i}^*(x - x_2) | = | e_{p_i}^*(x) - e_{p_i}^*(x_2) | = | 1 - \beta_{p_i} |$$

$$\leqq \| e_{p_i}^* \| \cdot \| x - x_2 \| \leqq \sqrt{p_i} \cdot \frac{\varepsilon}{\sqrt{m+1}} \leqq \varepsilon \qquad \text{for every } 1 \leqq i \leqq k.$$

Hence we have

$$\begin{aligned}
\| x_1 - x_2 \| &\geqq \left(\sum_{i=1}^{k-1} \frac{((\alpha_{p_{i+1}} - \alpha_{p_i}) + (\beta_{p_i} - \beta_{p_{i+1}}))^2}{p_{i+1} - p_i} \right)^{1/2} \\
&\geqq \left(\sum_{i=1}^{k-1} \frac{(| \alpha_{p_{i+1}} - \alpha_{p_i} | - 2\varepsilon)^2}{p_{i+1} - p_i} \right)^{1/2} \\
&= \left(\sum_{i=1}^{k-1} \frac{(\alpha_{p_{i+1}} - \alpha_{p_i})^2}{p_{i+1} - p_i} + 4\varepsilon^2 \sum_{i=1}^{k-1} \frac{1}{p_{i+1} - p_i} - 4\varepsilon \sum_{i=1}^{k-1} \frac{| \alpha_{p_{i+1}} - \alpha_{p_i} |}{p_{i+1} - p_i} \right)^{1/2} \\
&\geqq \left(\sum_{i=1}^{k-1} \frac{(\alpha_{p_{i+1}} - \alpha_{p_i})^2}{p_{i+1} - p_i} - 8\varepsilon \sqrt{m}(\sqrt{2} + \tfrac{1}{4}) \sum_{i=1}^{k-1} \frac{1}{p_{i+1} - p_i} \right)^{1/2} \\
&\geqq \left(\sum_{i=1}^{k-1} \frac{(\alpha_{p_{i+1}} - \alpha_{p_i})^2}{p_{i+1} - p_i} - 16\varepsilon \sqrt{m}(m+1) \right)^{1/2} \\
&= (\| x_1 \|^2 - 16\varepsilon \sqrt{m}(m+1))^{1/2} > \| x_1 \| - \tfrac{1}{4} \geqq \tfrac{3}{4} - \tfrac{1}{4} = \tfrac{1}{2},
\end{aligned}$$

a contradiction proving the proposition.

3.7. LEMMA. *Given* $x = \sum_{i=1}^{n} \lambda_i e_i$ *an element of* Y, *there exists* $m = m(x)$, $m \in \mathbf{N}$, *such that for all* $y \in Y$, *with* $\|y\| \leqq 1$, *and* $y = \sum_{i=n+m+1}^{s} c_i e_i$, *we have*

$$\|x + y\| = (\|x\|^2 + \|y\|^2)^{1/2}.$$

We need the following observation:

3.8. SUBLEMMA. *If* (a_n), (b_n), (c_n) *are three sequences of real numbers with* $b_n > 0$, $\lim_n |a_n| = +\infty = \lim_n b_n$, (c_n) *and* (a_n^2/b_n) *bounded sequences and* $\varepsilon > 0$, *then there is* $M \in \mathbf{N}$ *so that*

$$(*) \qquad \frac{(a_n + c_n)^2}{m + b_n} < \frac{a_n^2}{b_n} + \varepsilon \qquad \text{for } m \geqq M, \quad n = 0, 1, 2, \ldots .$$

PROOF OF SUBLEMMA 3.8. We notice first that if (*) holds for the pair (n, m), then actually (*) holds for all pairs (n, m'), with $m' \geqq m$.

The left-hand side of (*) has the form

$$\frac{a_n^2}{b_n} \cdot d_n, \qquad \text{where } d_n = \frac{\left(\frac{c_n}{a_n} + 1\right)^2}{\frac{m}{b_n} + 1}.$$

Clearly, (d_n) converges to 1, hence for $m = 1$ there exists n_0 such that for all $n > n_0$ we have

$$\frac{(c_n + a_n)^2}{1 + b_n} < \frac{a_n^2}{b_n} + \varepsilon.$$

It is obvious that for any $n \in \mathbf{N}$ there exists $m(n)$ such that the pair $(n, m(n))$ satisfies (*).

Hence if we set $M = \max\{m(1), \ldots, m(n_0), 1\}$, it is clear that M satisfies the desired properties.

PROOF OF LEMMA 3.7. Assume, on the contrary, that for an $x = \sum_{i=1}^{n} \lambda_i e_i$ there is no m satisfying the conclusion of the Lemma. Then for each $m \in \mathbf{N}$, there exists

$$y_m = \sum_{i=n+m+1}^{s_m} c_i^m e_i, \qquad \text{with } \|y_m\| \leqq 1,$$

and such that

$$(\|x\|^2 + \|y_m\|^2)^{1/2} < \|x + y_m\|.$$

It follows that $x + y_m$ attains its norm for some partition $p_1^m < p_2^m < \cdots < p_{k_m}^m$, so that there is an index $i_m < k_m$, with $p_{i_m}^m \leqq n$ and $p_{i_m+1}^m > n + m$. Hence

$$\frac{(y_m(p^m_{i+1})-x(p^m_{i_m}))^2}{p_{i_{m+1}}-p_{i_m}} > \frac{y_m(p_{i_{m+1}})^2}{p_{i_{m+1}}-(n+m)} + \frac{x(p^m_{i_m})^2}{n+1-p_{i_m}}.$$

This last inequality implies that

$$y_m(p_{i_{m+1}})\cdot x(p^m_{i_m}) \neq 0.$$

We set

$$\varepsilon = \min\left\{\frac{x(i)^2}{n+1-i}: 1 \leqq i \leqq n,\ x(i)\neq 0\right\},$$

$$a_m = y_m(p^m_{i_{m+1}}), \qquad b_m = p^m_{i_{m+1}} - (n+m),$$

$$c = \max\{x(i): 1 \leqq i \leqq n\}, \qquad \text{and}$$

$$c_m = \operatorname{sgn}(a_m)\cdot c \qquad \text{for } m \in \mathbf{N}.$$

Clearly we have

$$\frac{(c_m+a_m)^2}{m+b_m} > \frac{(y_m(p^m_{i_{m+1}})-x(p^m_{i_m}))^2}{p^m_{i_{m+1}}-p^m_{i_m}}$$

$$> \frac{a_m^2}{b_m} + \frac{x(p^m_{i_m})^2}{1+n-p^m_{i_m}} > \frac{a_m^2}{b_m} + \varepsilon.$$

CLAIM. The sequence $|a_m|$ tends to infinity.

(If not, there exists a subsequence $\{a_{m_l}\}_{l=1}^{\infty}$ which is bounded, say $|a_{m_l}| < d$ for $l = 1, 2, \ldots$; then obviously

$$\lim_l \frac{(c_{m_l}+a_{m_l})^2}{m_l+b_{m_l}} = 0,$$

hence there is l_0 with

$$\frac{(c_{m_{l_0}}+a_{m_{l_0}})^2}{m_{l_0}+b_{m_{l_0}}} < \varepsilon < \frac{a^2_{m_{l_0}}}{b_{m_{l_0}}} + \varepsilon,$$

a contradiction which proves the claim.)

Now we observe that the sequence (b_m) also tends to infinity, since $\|y_m\| \leqq 1$ and so the sequences (a_m), (b_m), (c_m) satisfy the assumptions of Sublemma 3.8; hence there exists an M such that for all $m > M$

$$\frac{(c_m+a_m)^2}{m+b_m} < \frac{a_m^2}{b_m} + \varepsilon,$$

a contradiction proving the Lemma.

3.9. PROPOSITION. *For every sequence* (x_n) *of blocks of* (e_n), $\|x_n\| \leqq 1$, *there is a subsequence* (x_{n_k}) *of* (x_n) *so that*

$$\left\| \sum_{i=1}^{k} x_{n_i} \right\| = \left(\sum_{i=1}^{k} \|x_{n_i}\|^2 \right)^{1/2} \qquad \textit{for } k = 1, 2, \ldots .$$

PROOF. We set $x_{n_1} = x_1$ and we assume that $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$ have been chosen so that for $k' \leqq k$ the finite sequence $x_{n_1}, \ldots, x_{n_{k'}}$ satisfies the conclusion. We set $y_k = x_{n_1} + \cdots + x_{n_k}$ and from Lemma 3.7 there exists $m_k \in \mathbf{N}$ satisfying:

$$\text{If } y_k = \sum_{j=1}^{s} c_j e_j, \quad \text{and } y \in Y \text{ with } \|y\| \leqq 1,$$

$$y = \sum_{j=s+m_{k+1}}^{s'} c_j e_j, \quad \text{then } \|y_k + y\| = (\|y_k\|^2 + \|y\|^2)^{1/2}.$$

Choose n_{k+1}, so that $x_{n_{k+1}} = \sum_{j=s_1}^{s_2} c_j e_j$, with $s_1 > s + m_k$. Then clearly the sequence $x_{n_1}, \ldots, x_{n_{k+1}}$ satisfies the inductive assumptions and the proof of the proposition is complete.

3.10. COROLLARY. *The space* X *does not contain isomorphically any* l_p, $1 \leqq p < \infty$, $p \neq 2$ *or* c_0.

PROOF. If some of these sequence spaces (l_p, $1 \leqq p < \infty$, $p \neq 2$, c_0) were isomorphically embedded into X, then there would exist sequence (x_n) in X equivalent to the usual basis of corresponding space (l_p, $1 \leqq p < \infty$, $p \neq 2$, c_0). Hence by Proposition 3.2, there would exist a sequence (y_n), consisting of blocks of (e_n), also equivalent to the basis of such space (l_p, $1 \leqq p < \infty$, $p \neq 2$, c_0). We then could get a subsequence $(y_{n_s})_{s=1}^{\infty}$, so that the conclusion of Proposition 3.9 would be satisfied. This would contradict the fact that the sequence $(y_{n_s})_{s=1}^{\infty}$ is equivalent to the basis of the original space (l_p, $1 \leqq p < \infty$, $p \neq 2$, c_0). The proof of the Corollary is now complete.

3.11. COROLLARY. *The space* X *does not admit an equivalent norm* $|||\cdot|||$ *so that* $(X, |||\cdot|||)$ *is a stable Banach space.*

PROOF. By a result of Guerre–Lapreste [6] (Theorem 1), if X is stable and it does not contain l^1, then X is reflexive. The Corollary now follows immediately from Corollary 3.10, and Proposition 3.6.

3.12. PROPOSITION. *The space* X *is weakly stable.*

PROOF. Assume that the space X is not weakly stable. Then it is easily seen that there are $\varepsilon > 0$, sequences (y_n), (x_m) and $x \in X$ such that

$$\text{weak-}\lim_k y_n = 0, \qquad \text{weak-}\lim_n x_n = x,$$

$$\lim_n \| y_n + z \| = \tau(z) \qquad \text{and}$$

$$\lim_n \| x_n + z \| = \sigma(z) \quad \text{for } z \in X, \qquad \text{and}$$

$$\lim_n \lim_m \| x_n + y_m \| - \lim_m \lim_n \| x_n + y_m \| \geqq \varepsilon.$$

Next we may assume (by Proposition 3.2), by passing if necessary to subsequences, that there are sequences $(\bar{y}_m)$, $(\overline{x_n - x})$ of blocks of (e_n) so that

$$\lim_m \| y_m - \bar{y}_m \| = 0 \quad \text{and} \quad \lim_n \|(x_n - x) - \overline{x_n - x}\| = 0.$$

We choose $x_0 \in X$, such that $\| x_0 - x \| < \varepsilon/4$ and $x_0 = \sum_{i=1}^k c_i e_i$. Then

$$\begin{aligned} | \, \| x_0 + \overline{x_n - x} + \bar{y}_m \| - \| x_n + y_m \| \, | &\leqq \| x_0 + \overline{x_n - x} + \bar{y}_m - (x_n + y_m) \| \\ &\leqq \|(x_0 - x)\| + \| \overline{x_n - x} - (x_n - x) \| + \| \bar{y}_m - y_m \|, \end{aligned}$$

therefore there are n_0, m_0 so that for any $m > m_0$, $n > n_0$ we have

$$(*) \qquad \| x + \overline{x_n - x} + \bar{y}_m \| - \| x_n + y_m \| < \varepsilon/3.$$

Applying Lemma 3.7, we get

$$\lim_m \| x_0 + \overline{x_n - x} + \bar{y}_m \| = (\| x_0 + \overline{x_n - x} \|^2 + \tau(0)^2)^{1/2}$$

and

$$\lim_n (\| x_0 + \overline{x_n - x} \|^2 + \tau(0)^2)^{1/2} = (\| x_0 \|^2 + \sigma(-x)^2 + \tau(0)^2)^{1/2};$$

for the same reasons we have

$$\lim_m \lim_n \| x_0 + \overline{x_n - x} + \bar{y}_m \| = (\| x_0 \|^2 + \sigma(-x) + \tau(0)^2)^{1/2}.$$

Therefore from $(*)$ we get

$$\lim_n \lim_m \| x_n + y_m \| - \lim_m \lim_n \| x_n + y_m \| < 2\varepsilon/3,$$

a contradiction.

3.13. COROLLARY. *For every $\varepsilon > 0$, every infinite dimensional subspace of X contains $(1+\varepsilon)$-isomorphically a copy of l^2.*

PROOF. This follows immediately from Theorem 1.2, Proposition 3.12, and Corollary 3.10.

Quite analogously we can prove the following:

3.14. THEOREM. *Let $1 < p < \infty$. There is an infinite dimensional Banach space X such that*

(a) *X is weakly stable,*

(b) *X does not admit an equivalent stable norm (hence X is not reflexive), and*

(c) *for every $\varepsilon > 0$, l^p embeds $(1+\varepsilon)$-isomorphically in every infinite dimensional subspace of X.*

3.15. QUESTION. We do not know if it is possible to prove the existence of a Banach space X satisfying the properties of Theorem 3.14 with $p = 1$. Our example does not help in that direction.

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